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THE MATHEMATICS TEACHER

VOLUME XX MARCH, 1927 NUMBER 3

SOME OF EUCLID'S ALGEBRA

By GEORGE W. EVANS

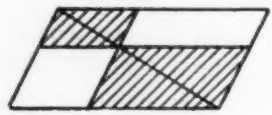
If Euclid had called his *Elements* the "Elements of Geometry," that name would have been a misfit, for it was geometry only to a minor extent. It included the elements of all the mathematical knowledge of his time, as well as of much in times far distant. Many things had been separately known, the achievements of different wise individuals following paths apparently diverging. He was able to see the unity in these scattered treasures, and had the skill to frame a system embodying that unity. He found it in number.

Numbers for his geometry and geometry for his numbers. But the numbers that were for him the clue to the maze of exact knowledge then available were not known to him by any inclusive name. His idea was wide but vague. His definite word, number, was narrow; it was saved for integers. All other numbers were covered up in his diagrams: there was the line that, by a clumsy substitute for our term "rational fraction," he called a "part" or "parts" of another line, or the line that had to some other line the ratio which one integer has to another; and there was the line that had not to some other line the ratio which one integer has to another. It is not surprising that many of his editors were unable to see what his pearls were strung on.

The Notation for Greek Algebra.—The lengths of lines with him might represent numbers, or quantities in general; the product of two numbers was the area of a rectangle. To add one number to another was to annex one distance to another, in line; to subtract one number from a larger number was to cut off a distance from a larger distance. Purely algebraic conclusions, stated in these geometric terms, had to be left without further comment. They might be about integers, or fractions, or irrational numbers; and even if there was a vague idea including these three, there was no word for it.

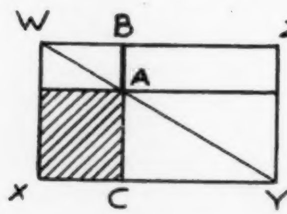
The Theorem of the Gnomon.—A fundamentally important part of Euclid's geometrical algebra was a certain group of theorems about parallelograms; they are as a whole unknown to American high schools. The first of this group is Proposition 43 of Book I, the "Theorem of the Gnomon":

In any parallelogram the complements of parallelograms about the diagonal are equal to one another.



In the diagram the shaded figures are the "parallelograms about the diagonal"; the unshaded figures are the complements. Either of the parallelograms about the diagonal, together with both complements, forms what Euclid calls a gnomon.

In the next proposition he applies this theorem to construct, on a given base, a rectangle equal to a given rectangle. In our diagram the shaded rectangle is given, and the line AB is given



as base. We have only to complete the rectangle $XCBW$, extend the diagonal WA until it meets the base as at Y, and then complete the rectangle $WXYZ$. Then the shaded rectangle is one of the complements, and the other complement will be the rectangle we were required to construct.

We have here a method for constructing the quotient obtained by dividing the product of two numbers by another number. It will serve also to transform two rectangles into rectangles of the same height, even of unit height, and so to obtain lines that will represent the same numbers as the areas do; and since a number represented by a line may also be represented by a rectangle of unit altitude on that line as base, the Greeks could divide as well as multiply any number by any other number,

without being restricted to apparent geometrical dimensions, as without this powerful theorem they would have been restricted.

These theorems are stated by Euclid more generally, referring to parallelograms with a given angle, instead of to rectangles. The other theorems of the group, which constitutes as a whole the doctrine of "application of areas," depend on the theory of proportion.

The Formulas of Book II.—Book II presents to us "one of the ways in which the Greeks before Euclid represented algebraic operations,"[1] the other way being the use of proportions. The first ten theorems, expressed in our symbolism, are as follows:

1. $a(b + c + d + e + \cdots) = ab + ac + ad + ae + \cdots.$
2. $(a + b)a + (a + b)b = (a + b)^2.$
3. $(a + b)a = a^2 + ab.$
4. $(a + b)^2 = a^2 + b^2 + 2ab.$
5. $(a + b)(a - b) + b^2 = a^2.$
6. $(2a + b)b + a^2 = (a + b)^2.$
7. $a^2 + b^2 = 2ab + (a - b)^2.$
8. $4(a + b)a + b^2 = (2a + b)^2.$
9. $(a + b)^2 + (a - b)^2 = 2a^2 + 2b^2.$
10. $(2a + b)^2 + b^2 = 2(a + b)^2 + 2a^2.$

It seems obvious that the first four are logically successive, the first implying the second and the third, and those two the fourth. Euclid, however, does not so prove them. Every one of these ten theorems, though they are by no means independent, he proves without referring to any other theorem of this Book. He was furnishing the proofs for a set of facts well known and commonly used as rules of computation. He made them here a part of the scientific system of which Book I was the comprehensive beginning; apparently he also used them to show the effectiveness of his method. Although the general principle of distributiveness enters of course into all of Euclid's algebra, he does not again refer to the first three theorems; or to the eighth, ninth, or tenth. Where the others are referred to, they are used to establish other numerical relations—for example, in Book X, in the discussion of irrational lines.

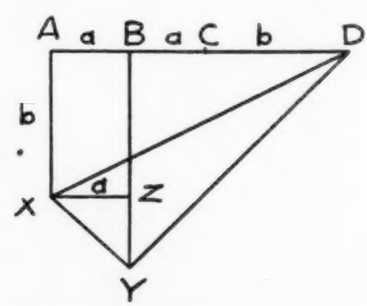
Since the theorems from which these formulas are derived are

[1] H. G. Zeuthen, *Sur l'Origine de l'Algèbre* (Copenhagen, 1919), p. 22.

expressed in terms of the rectilinear figures, we can get a variety of algebraic interpretations merely by changing our lettering. Thus, in the sixth, if we now call a the line that we previously have called $a+b$, and b the line that we have called a, the theorem gives the same formula as the fifth; and in the ninth, if we change the name of the line b to a, and the name of the line a to $a+b$, we get the same formula as the tenth.

A Formula of Approximation.—This last formula, Proposition 10, is famous as the first rule in history for finding successively closer approximations of an irrational number. Its proof is as follows:

Let $AB=BC=a$, and $CD=b$, all in line. At A draw $AX \perp AD$, and make $AX=b$. Then, AXD being a right tri-



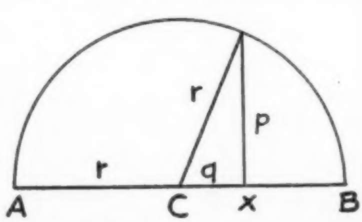
angle, $(2a+b)^2+b^2$ will be the square of XD. Then at B draw $BY \perp AD$, and make $BY=b+a$. Draw $XZ \parallel AD$, cutting BY as at Z. Then $XZ=ZY=a$, and the square of XY $=2a^2$. The square of $DY=2(a+b)^2$. Since the right triangles ADX and DYX have the same hypotenuse, we have by the Pythagorean theorem the equation: $(2a+b)^2+b^2=2a^2+2(a+b)^2$.

In using the formula, take any approximation to $\sqrt{2}$, even a wild one; for example, let a be the length of the side of a square, and b the length roughly guessed for the diagonal. If these numbers are expressed in figures without radical signs or any other adulteration, it is not possible that $b^2=2a^2$. There is what we may call an error of area. Now the formula enables us to get two larger numbers, $2a+b$ and $a+b$, with the same error of area; that is, they are such that $(2a+b)^2-2(a+b)^2=2a^2-b^2$.

If $7/5$ is taken for the first approximation to $\sqrt{2}$, the error of area is $7^2 - 2(5^2) = -1$; in the formula, $2a + b = 2 \times 5 + 7 = 17$, and $a + b = 12$. Now, with these larger numbers, the error of area is the same as when we used the numbers 5 and 7, for $289 - 2(144) = 1$. The undeniable truth of the formula makes it certain that no matter how many times we pass to a larger pair of numbers, the error of area will be unchanged in amount. We need, then, only patience and a pencil to get as close an approximation as could ever be needed.

The eleventh proposition, which gives the construction for dividing a straight line in extreme and mean ratio, is generally spoken of as a graphic solution of the equation $a(a - x) = x^2$; and the twelfth and thirteenth prove "the law of cosines."

The fourteenth proposition, which closes Book II, shows how to construct a square equal to a given rectangle; that is, it solves graphically the equation $x^2 = ab$. It finds the square root of the number represented by the rectangle. The construction of this problem is as with us, but the proof, instead of depending on similar triangles, uses the Pythagorean theorem. If two unequal sides of the rectangle, as x and y, are laid off in line, say $AX = x$ and $XB = y$, then you describe a semicircle on $x + y$



as a diameter, and let C be the center of the circle. Draw the half chord $p \perp AB$ at X. If r is the radius of the circle and q the distance CX, you have, from Proposition 5, $(r + q)\ (r - q) = r^2 - q^2$, and $r + q = x$, $r - q = y$, and $r^2 - q^2 = p^2$, so that you have proved that $xy = p^2$. That's it.

Geometry for Algebra's Sake.—This Pythagorean proposition is the 47th of the first Book. It is about a geometrical figure, but is algebraic in content and in purpose. It is used as a formula in Book II and in the other algebraic Books; it occurs often in the discussion of irrationals. Wherever it occurs, even to prove a geometrical fact like perpendicularity, it is an alge-

braic formula.[2] The old theorem by itself furnished to the Greeks a sort of algebraic game of solitaire. There was a besetting temptation to find the integers that would satisfy the equation $x^2 + y^2 = z^2$. Pythagoras had one rule for them, and Plato another; Euclid gives them both,—somewhat disguised; and he gives one of his own.[3]

Euclid found use for the Pythagorean theorem in the last six of the fourteen theorems in Book II. It is supposed that the earlier proofs of that familiar proposition depended on the principle of similarity. Euclid is credited with the invention of the "mouse-trap," the traditional proof which appears in Book I, and which made possible the early use of the formula $x^2 + y^2 = z^2$. Because of it he was enabled to complete almost at the start his algebraic foundations, and equip himself for those laborious investigations that we with our handier symbols can of course carry out so much more easily and so much more completely.

The last six of the 48 propositions of Book I can fairly be called algebraic in purpose if not in content; and yet, where any of these or of the other algebraic propositions in the *Elements* have survived in our school books, we have been expected not to see any algebra in them.

The Commonest Equations in Geometry.—The subject of proportion, to which Euclid's fifth Book is devoted, was never really beloved in our teaching of geometry; now it has been sent away to the algebra class, which is more indulgent. The subject is of course algebraic, but it is no more so now than it ever was; we are merely applying to it a clearer and more adaptable notation, instead of using Euclid's confusing lines. We ignore, however, as always in our algebra, the little difficulty of incommensurables, which Euclid has successfully dealt with by means of his definitions. He had the advantage of using his geometry, like an egg in the coffee pot, to clarify his study of number.

Because his algebra was expressed only in his vague geometrical notation, and for that reason only, Euclid had to invent names for the different transformations of the standard equation that he called a proportion. Most of them are forgotten. They

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[2] See *e.g.*, *Euclid's Elements*, XI, 35.

[3] Heath's *Euclid*, Vol. 1, p. 360.

ought to be. They did give to these simple and ordinary equations a peculiar separateness and dignity, creating a sort of Brahmin caste, so that for a long time we hardly ventured to recognize them. You must not on that account let anyone tell you that a proportion was not an equation to Euclid. It is true that his customary form of statement, "as one thing is to some other, so is this to that," does not contain the words "is equal to"; but against that you can balance his definition of proportion, which was explicitly a definition of equal ratios, and the occasional quite casual use of the statement that one ratio is the same as another.[4]

Aside from the extraordinary mathematical insight shown in the definitions and general treatment of proportion, the most striking thing in Euclid's algebra is the so-called "application of areas." The central theorems of this doctrine are the twenty-eighth and twenty-ninth of Book VI.

The Powerful Parallelograms.—In simplified form the twenty-eighth is the problem to apply to a given straight line a rectangle equal to a given polygon, "and deficient by a rectangle similar to a given one."

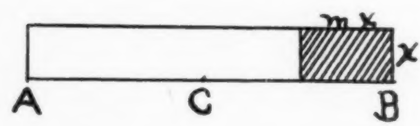
The meaning of the phrase "deficient by a rectangle," etc., is made clear by this figure. *AB* is the given line, and the unshaded rectangle is applied to it, but fails to cover it; it lacks the shaded rectangle. We are to make the blank rectangle equal to the given polygon, and the shaded one similar to the given rectangle; in other words, the ratio of its sides is given. The blank rectangle is called "deficient," not because it falls short

of the required area—for it does not—but because it does not cover the entire line *AB*.

[4] *E.g.*, Book X, Prop. 5. With the different transformations that he put on his list, he was able to do with his proportions almost everything that we do with linear equations. The oldest inhabitant in our most ancient city can no doubt remember problems that he himself used to solve by the Rule of Three—some of the very problems whose cadavers are now dissected by algebra. It was a survival of an antiquated method, the same method that was used, for example, by Galileo, before modern algebraic notation had come into use.

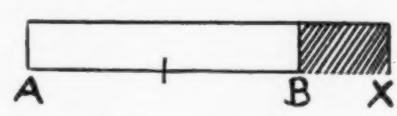
Quadratics Solved Graphically.—The connection of this problem and the next with algebra is one of the surprises that we American teachers meet with in the study of Euclid. Each contains the graphic solution of a quadratic equation, and is used in the theoretical discussion of quadratic equations by Euclid himself and by his immediate successors.

Thus, in the twenty-eighth, if C is the middle point of AB, call $AC = a = CB$, call the altitude of both rectangles x, and call the base of the shaded rectangle mx. The ratio m is determined by the required similarity. The area of the blank rectangle is therefore $(2a - mx)x$, and when we make it equal to the given polygon, whose area we could call S, we have the equation S



$= 2ax - mx^2$. Then, when we show how to construct this rectangle, we show how to solve the equation; and it is even claimed that a numerical solution can be guided by this construction.

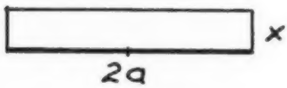
The twenty-ninth is the problem to apply to a given straight line a rectangle equal to a given polygon, and "exceeding" by a rectangle similar to a given one. Here the rectangle required to be equal to the given polygon is "applied" to the given line AB, but projects beyond the end of AB, as indicated by the dotted line BX. To the extent indicated by the shaded region,



it exceeds the rectangle that would exactly cover AB. Now if, as required, the entire rectangle reaching from A to X is equal to the given area S, then we have the equation $S = 2ax + mx^2$.

Echoes Reaching Us.—If we now take y^2 instead of S to represent the given area, and remember that by the "theorem of the gnomon" we can construct a rectangle that will just cover AB, we have three diagrams and three equations corresponding to them as follows:

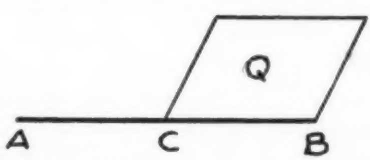
$$y^2 = 2ax$$



$$y^2 = 2ax - mx^2$$

$$y^2 = 2ax + mx^2$$

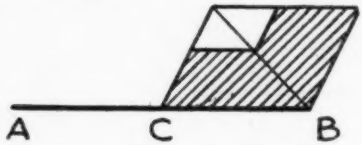
The Greek names of these three constructions are, respectively, *parabole, elleipsis,* and *hyperbole.* The three equations as written here are the precise forms in which we customarily express certain characteristics of the conic sections. It was because of these equations, obtained in this way and expressed in the clumsy Greek algebra, that Apollonius bestowed on the curves the familiar names of the Euclidean constructions; and those names have to-day the meaning that Apollonius gave them.

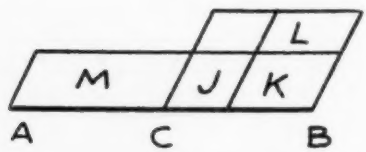


Ancient Graphics.—Euclid made these two propositions refer not to rectangles but to any parallelograms. Aside from the fact that he liked generality anyway, the admission of oblique parallelograms gave to their application in geometry the same advantage that the use of oblique coordinates does with us. The construction and proof are just as simple with the parallelogram in general as with the rectangle. For the twenty-eighth you bisect the given line, and on half of it construct a parallelogram similar to the given one. Call it Q.

Then construct, "about the diagonal," a parallelogram similar to Q and equal to $Q - S$. The shaded gnomon then will be equal to S. Letter its three parts J, K, and L, and construct, on the line AC, a parallelogram congruent with $J + K$; call it M.

$M + J = J + K + L$ (since $J = L$), which is, as we saw, equal to S. Then $M + J$ is the required parallelogram, and its short side is the value of x.





The proof of the twenty-ninth is substantially the same, only the parallelogram $Q + S$ is formed, instead of $Q - S$, and the gnomon equal to S is outside of Q, the required parallelogram being formed below the line AB.

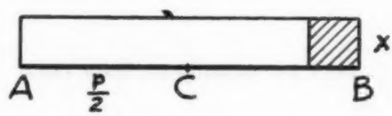
The Antique Discriminant.—It indicates the thoroughness of Euclid's treatment of quadratics that he has in Book VI, Proposition 27, a proof that in the equation $S = 2ax - mx^2$ the value of S cannot exceed a^2/m. It is evident that in the figure last discussed, since S is the area of the gnomon, it must be less than the parallelogram Q by the parallelogram above on the left. If we sacrifice the generality of oblique coordinates, we get rid of the sine coefficient and Q becomes a well-behaved rectangle equal to a^2/m. This restrictive condition naturally precedes the proposition which obtains the solution of the equation. The solution is a straight line, and not necessarily a rational straight line, so that this proposition is a condition for the existence of a real root. It is the same as our algebraic condition, namely, that since $x = a/m \pm \sqrt{(a^2/m^2) - (S/m)}$, therefore S/m cannot exceed a^2/m^2.

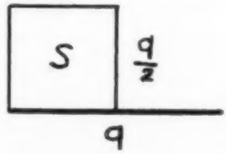
He had also a criterion for the rationality of the roots. In Book X, Proposition 17 is the following theorem:

If there be two unequal straight lines, p > q, and if to p there be applied a rectangle equal to ¼q² and deficient by a square figure, then, whenever p is thereby divided into commensurable parts, p² exceeds q² by the square on a straight line commensurable with p.

He has himself abandoned here not only his oblique parallelo-

grams, as we did, but also what was in our figures the coefficient m, the ratio of the sides of the rectangle. His theorem amounts to saying that in the equation $x(p-x)=S$, if x is to be commensurable with p, then p^2-4S must be equal to $m^2/n^2 \times p^2$, when m and n are integers. Now if p is itself rational, any perfect square is of the form $m^2/n^2 \times p^2$; so that the theorem actually says that if the roots of $x^2-px+S=0$ are rational, then p^2-4S is a perfect square.





In the proof of this theorem there is an illustration of the flexibility of geometric symbols in algebra. In the formula of Proposition 5, Book II, which we have written $(a+b)(a-b)+b^2=a^2$, if we let $a=p/2$, and $b=p/2-x$, we have at once

$$(p-x)x+\left(\frac{p}{2}-x\right)^2=\frac{p^2}{4}.$$

Now by construction $(p-x)x=q^2/4$; and therefore

$$q^2/4+(p/2-x)^2=p^2/4, \text{ or } p^2-q^2=(p-2x)^2.$$

By hypothesis x is commensurable with p; then $p-2x$ is commensurable with p, and p^2-q^2, which is equal to $(p-2x)^2$, is "the square on a straight line commensurable in length" with p. There you are.

What he has proved is in our notation this: In the equation $x^2-px+S=0$, if the roots are rational, p^2-4S is a perfect square. In the second part under the same caption he proves that if p^2-4S is a perfect square the roots are rational. Then in the next proposition he gives wholly independent proofs showing that if the roots are not rational p^2-4S is not a perfect square, and that if p^2-4S is not a perfect square the roots are not rational. The last two theorems are of course immediately

implied by the first two. One would like to think that he wanted his students to remember the irrational root, and took this method of rubbing it in; but Heath says that Euclid did not permit himself such inferences, and De Morgan says that he did not know any better. You must choose between loyalty and logic.

The tenth Book contains a complete and detailed description of all the kinds of irrationals that can be roots either of the equation $x^2 - 2ax + S = 0$, or of the equation $x^4 - 2ax^2 + S = 0$. There are in all 115 propositions, of which one or two were probably inserted by busybodies long forgotten. It is the longest of the thirteen books, and is remarkable for its finished perfection.[5] It is purely algebraic. The irrationals discussed are described by classes, beginning with the roots of the biquadratic. The following list gives numerical samples of each class, in Euclid's order, with the numbers of the propositions in which it is shown how the lines that may represent them can be constructed, and with the equations (in our notation) for which the first column gives solutions. Euclid's notation would be the "application" of an area to a straight line.

The List of Irrationals

The first of this list is defined in Proposition 21 as follows:

The rectangle contained by straight lines commensurable in square only is irrational, and the side of the square equal to it is irrational. Let [the latter] *be called medial.*

The medial then is what we call the geometric mean between two numbers like 3 and $\sqrt{2}$, or between $\sqrt{5}$ and $\sqrt{3}$. It would be a number like $\sqrt[4]{3\sqrt{2}}$, or $\sqrt[4]{15}$, or $\sqrt[4]{3}$. The square of a medial is a "medial area," for example $3\sqrt{2}$ or $\sqrt{15}$, or $\sqrt{3}$. The expression $2 + \sqrt{3}$ is sometimes called the sum of "two straight lines commensurable in square only," and sometimes the sum of "a rational and a medial area." When the same number can have its name changed in this high-handed way, the purely algebraic character of the geometric symbols is obvious.

The medial is the solution of an equation of the form $x^4 = A$, where A is not a square number. Here are the rest:

[5] Heath's *Hist. Greek Math.*, p. 411.

		Subclasses of Binomials		Equations
$\sqrt{6} + 1$ (Binomial)	36	$7 + 2\sqrt{6}$ (First)	48	$x^4 - 14x^2 + 25 = 0$ $x^4 - 4x^2\sqrt{6} - 25 = 0$
$\sqrt[4]{2}(3 + \sqrt{2})$	37	$3\sqrt{2} + 4$ (Second)	49	$x^4 - 6x^2\sqrt{2} + 2 = 0$ $x^4 - 8x^2 - 2 = 0$
$\sqrt[4]{2}(\sqrt{6} + 1)$	38	$7\sqrt{2} + 4\sqrt{3}$ (Third)	50	$x^4 - 14x^2\sqrt{2} + 50 = 0$ $x^4 - 8x^2\sqrt{3} - 50 = 0$
$\sqrt{\frac{7 + \sqrt{39}}{2}} + \sqrt{\frac{7 - \sqrt{39}}{2}}$	39	$7 + \sqrt{10}$ (Fourth)	51	$x^4 - 14x^2 + 39 = 0$ $x^4 - 2x^2\sqrt{10} - 39 = 0$
$\sqrt[4]{3}\left(\sqrt{\frac{5 + \sqrt{22}}{2}} + \sqrt{\frac{5 - \sqrt{22}}{2}}\right)$	40	$5\sqrt{3} + 3$ (Fifth)	52	$x^4 - 10x^2\sqrt{3} + 66 = 0$ $x^4 - 6x^2 - 66 = 0$
$\sqrt[4]{2}\left(\sqrt{\frac{7 + \sqrt{39}}{2}} + \sqrt{\frac{7 - \sqrt{39}}{2}}\right)$	41	$7\sqrt{2} + 2\sqrt{5}$ (Sixth)	53	$x^4 - 14x^2\sqrt{2} + 7 = 0$ $x^4 - 4x^2\sqrt{5} - 78 = 0$
		Subclasses of Apotomes		
$\sqrt{6} + 1$ (Apotome)	73	$7 - 2\sqrt{6}$ (First)	85	Same as on first line
$\sqrt[4]{2}(3 - \sqrt{2})$	74	$3\sqrt{2} - 4$ (Second)	86	Same as on second line
$\sqrt[4]{2}(\sqrt{6} - 1)$	75	$7\sqrt{2} - 4\sqrt{3}$ (Third)	87	Same as on third line
$\sqrt{\frac{7 + \sqrt{39}}{2}} - \sqrt{\frac{7 - \sqrt{39}}{2}}$	76	$7 - \sqrt{10}$ (Fourth)	88	Same as on fourth line
$\sqrt[4]{3}\left(\sqrt{\frac{5 + \sqrt{22}}{2}} - \sqrt{\frac{5 - \sqrt{22}}{2}}\right)$	77	$5\sqrt{3} - 3$ (Fifth)	89	Same as on fifth line
$\sqrt[4]{2}\left(\sqrt{\frac{7 + \sqrt{39}}{2}} - \sqrt{\frac{7 - \sqrt{39}}{2}}\right)$	78	$7\sqrt{2} - 2\sqrt{5}$ (Sixth)	90	Same as on sixth line

Of the remaining propositions in this Book, two will serve as simple illustrations of Euclid's method. Proposition 54 is as follows:

If an area be contained by a rational straight line and a first binomial, the side of the area is the irrational straight line called binomial.

The words "area contained" mean of course a rectangle, and "the side of the area" is the side of a square equal to the rec-

[6] The second equation in each case is of the form $x^2 + px - q = 0$. Euclid gives no construction for solving the quadradic $x^2 - 2ax - b = 0$.

tangle, the square root of the area. The rational straight line is any arbitrarily specified straight line commensurable either "in length" or "in square" with the unit. It may of course be the unit. If it is, the proposition shows that the square root of a first binomial is itself some one of our six binomials. If it is not, the proposition serves in the discussion of more complicated figures—generality is Euclid's habit.

Proposition 60 says: *The square on a binomial straight line applied to a rational straight line produces as breadth the first binomial.* That means that if for example you construct a rectangle, on a unit base, equal to the square on a binomial, the altitude of the rectangle will be a first binomial.[7] The altitude in a rectangle of unit base is numerically equal to the area; consequently we have a line to represent the square of any irrational number like $3 + \sqrt{2}$ or $\sqrt{3} + \sqrt{2}$, or $\sqrt{3} + 1$; just as in theorem 54 we had a line to represent the square root of such things as $7 + 2\sqrt{6}$, or $3 + \sqrt{5}$, or $2 + \sqrt{3}$.

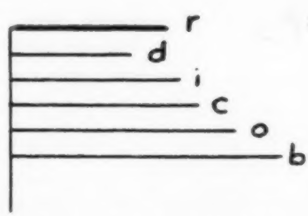
It is further proved that the square of any apotome is a first apctome, and the square root of a first apotome is some kind of an apotome; in general, that the square of any expression in the first column of our table would be one of the same class as the expression in the same line in the second column; and the square root of any expression in the second column would be like the one on the same line in the first column. These proofs apply to classes, not to individual irrationals. It is a more general, a really algebraic treatment of the subject—our way deals with particular numbers and in contrast with his might be called arithmetical.

Other theorems in the Book show that a line commensurable with any one of these irrationals is itself an irrational of the same type; that the larger terms in two such irrationals are commensurable, and the smaller terms likewise; and that an irrational line of any one of these types can be divided into its incommensurable parts in only one way, so that for example if $a + \sqrt{b} = x + \sqrt{y}$, then $a = x$ and $b = y$.

Edges of Regular Solids.—Books VII, VIII, and IX are wholly devoted to a theory of integers, every integer being represented by the length of a straight line referred to some tem-

[7] In a first binomial the first term is rational (in Euclid's sense) and the difference of the terms is a square number.

porary unit; Book XII is a discussion of the proportionality of circles and solids; and Book XIII is an investigation, complicated enough, of the numerical characteristics of the five regular polyedra. In this last Book he finds that for a regular pyramid inscribed in a sphere of radius r, the edge $t = \frac{2}{3}r\sqrt{6}$; for an octahedron, $o = r\sqrt{2}$; for a cube, $c = \frac{2}{3}r\sqrt{3}$; for a dodecahedron d is an apotome, and for an icosahedron i is the square root of a fourth apotome. Then in the last proposition he constructs and "sets out" the edges of these five solids and compares them with the radius of the sphere in which they are all inscribed. It is a graph, showing their relative lengths. The accompanying diagram shows these lines as we would arrange them, the lengths being taken from the diagram in Heath's *Euclid*. They are marked with the initial letters of the figures they belong to.



Of the thirteen Books credited to Euclid, six are entirely algebraic, and four of the others are to some extent algebraic. Again, some of the algebraic theorems enter fundamentally into the geometry, and some of the geometrical theorems into the algebra. Even to-day, with all the expurgations that have been effected, and though it may be forgotten or denied and concealed in clumsy and antiquated notation, there is a considerable residue of algebra in the remnant of Euclid that is included in our textbooks. In dealing with it we can make our teaching clearer and more comprehensible, if not more comprehensive, by using the notation that made possible the wide spread of mathematical knowledge in modern times.

A NUMBER OF THINGS FOR BEGINNERS IN GEOMETRY[1]

By VESTA A. RICHMOND

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Among the aims of mathematical instruction given by the National Committee are the cultural aims, and listed under these is "appreciation of beauty in the geometrical forms of nature, art, and industry." Young, in *The Teaching of Mathematics,* says "subjects have value on account of the information they impart," and another report says ". . . on the other hand the power of viewing mathematically the world of phenomena surrounding us should be developed as highly as possible," and still another writer says "a preliminary test of the worthwhileness of a course in mathematics is the contribution that such a course makes in the direction of greater intelligence in the other subjects of the curriculum."

Sometimes we forget these aims or leave them to the various mathematical clubs to develop, but usually the membership of these clubs is restricted to the upper classes and to pupils who are maintaining a certain high average in their mathematical work. The *broadening* influences, then, of mathematics, and the interest aroused by venturing into fields which tie up the mathematics with other lines of thought and work are therefore lost to a great many who would perhaps particularly profit by them. If the beginner in geometry can be brought to realize by individual experimentation that geometrical forms and mathematical relationships exist *everywhere* in the world about him, a new pleasure in the subject, and a new wonder and admiration for it, and a new correlation with many of his other subjects are developed. There are innumerable fields to call upon; in the home life, the geometric designs of floor coverings, from the patterns of linoleum to the beautifully worked out designs of the oriental rugs; in play, the designs of games and the forma-

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[1] Presented at the Annual Meeting of The Association of Teachers of Mathematics in New England, December 4, 1926.

tion of certain puzzles; in nature, plant life; in science, the forms of crystals; in art, the principles of design, and of static and dynamic symmetry; in architecture, the Gothic, Persian, Segmental arches, etc.; in history, the expression of art of primitive people, and so on.

The later years of our high school course in the classical curriculum, at least, are largely—whether we like it or not—college preparatory, and it is not yet an easy task to prepare pupils for college and at the same time educate them, but the tenth school year offers the opportunity and so to-day I am going to tell you of some of the work my pupils have been doing as "extra" or "outside" work in the beginning course in geometry. To think of a suitable title to call the work has been more or less of a problem: whether or not it is a series of "projects" must be left to the psychologist, whether or not it may even claim the title of geometry exercises is a question; it is just a "number of things" we have been doing to bring geometry as close to the life of the pupil as possible, to broaden his horizon by giving him a glimpse of many fields outside the classroom, to bring out artistic or constructive abilities; in short, to teach him to enjoy *living* geometry.

Our plan is very simple. At the beginning of each week I briefly outline what I want them to do for that week's "outside work"—for this is all done in addition to their regular assignments. It is to be done at any time during the week, it is planned for their odd minutes. It is ordinarily not compulsory and as "extra work" it receives no particular mark with the exception that an unusually fine piece of work receives extra credit. In this perhaps we are following the idea of the New Education that work done under compulsion lacks wholeheartedness—and the response of the pupils has been most happy. As a rule they like to do it, and some spend hours on it—extra hours.

The first assignment gave a connection with what they already knew of geometry. Some magazine picture or advertisement (preferably colored) was to be cut out and mounted on a piece of paper. Underneath it were to be listed all the geometrical objects the pupil could find in the picture. The majority found from 8 to 10: rectangle, rug; square, table; circle, light-dome; etc., but one girl, from the picture of a Ford coach, found 25 different geometrical figures showing a fairly wide geometrical

vocabulary, and giving us a splendid opportunity to discuss the new words which she had used and which were unfamiliar to the others. In a remarkably short time we had on the blackboard a very workable list of geometric terms, each member of the class being more or less delighted to contribute a new word to the growing list. It was a very easy matter then to show how *definite* a definition must be, and the demand for clearness and exactness.

This project was a connection with the present, the next was with the past, and this was due two days later. The class was divided into three sections, each of which was to look up interesting facts about Pythagoras, Plato, or Euclid. The anecdotes which were told were most entertaining—the secret society of Pythagoras and a story of a member of the society, who, while away from his friends, was taken ill and died. The innkeeper at his request, hung the emblem of the society, a pentagon, where it was seen by a member weeks later, and he promptly paid the innkeeper for his kindness. There was also the story of the punishment by death for revealing the secrets or discoveries of the school, and of the possible death by drowning of one member who, elated by his discovery of "the sphere with twelve pentagons," boasted of it to a friend outside the society; Plato's first college entrance requirement, "Let no one enter here who does not know geometry," and his famous remark, "God eternally geometrizes," and Euclid's equally well-known remark in answer to Ptolemy's question, "There is no royal road to geometry."

Our third problem was to find examples in nature which illustrated the truth of the statement, "God eternally geometrizes." The results were a joy to the eye and a revelation of the ingenuity of the pupils. Booklets of leaves, showing the various types of symmetry, and the square stem of the mint plant; star-fish, sand-dollars, shells and stones; flowers illustrating all kinds of polygons; berries and grapes for beautiful spheres; pine cones, honeycombs, symmetrical trees, snowflake patterns, etc. The list is almost unlimited.

This nature work might well be carried on for several weeks right here if one wished to keep the nature work together, but to correlate the class work with this extra work it seemed better to introduce some construction problems, since the regular class work had been that of using rulers and compasses, bisecting lines,

angles, etc., and in order to further the pupils' facility with these instruments our next venture was to draw an original design or copy some design using merely the rulers and compasses. The interest and competition were keen here. From the art department we borrowed several large patterns and designs which were posted on the bulletin board. Some of these were copied, and the more difficult the design, the more there were who did it. The original designs were a joy, too, a gay little girl with billowy skirts selling balloons, fairy ships with sails spread wide, etc.—and—the moral to the tale, the pupils gained much in dexterity with the tools of geometry. It is much more interesting to make something beautiful by bisecting lines and angles and arcs than to merely do these things as geometric exercises. Cubist art, so popular a few years ago, was, after all, merely effects made with straight lines.

For the next week we did further work in construction by making the regular polyedrons. Patterns were placed on the board for the simpler ones, and if the pupils wished to make the more difficult ones they were referred to a solid geometry textbook. It was a delight to find how many wanted to do the dodecahedron. To be sure, it was the Waterloo of many, and one boy told me that no wonder the follower of Pythagoras who first discovered this "twelve faced sphere" was so overjoyed he *had* to tell his friends, even on penalty of death; he didn't blame him! These constructions were also a splendid lesson in accuracy, for if the polygons were not accurately constructed they would not fit to make the "perfect" solid. By leaving one of the sides of the octahedrons open we discovered that we could make an attractive candy box, and one class made over a hundred of these, which were filled with candy and sold at the annual Mathematics Club entertainment. Other pupils made beautiful electric light shades of the regular polyedrons, cutting out the faces and lining them with various tints and colors of tissue paper.

Our class work was now taking us into the measurement and size of angles, so this led us to the construction of simple transit instruments. Again, the ingenuity of the pupils went beyond my imagination. Three different boys rigged up instruments on tripods; one that was particularly clever was made of "Erector" materials, another hinged his on a block of wood so that it could be swung up vertically or lowered horizontally. One of the

girls used her jig-saw to cut from wood the circle which made her base and the quarter circle which fitted into this base and formed her vertical plane as well as serving as a pointer for the horizontal, the pointer for the vertical plane being an old umbrella rib fastened on with a piece of an old gas fixture! Another mounted two well-divided circles on heavy pieces of corrugated cardboard which he hinged at one side, giving, when opened up, the two planes. There were as many different crude transit instruments as there were members to the class but each person had the practice of dividing a circle into degrees and becoming familiar with the methods of angle measurements.

The next week's work was to measure some irregular pieces of land, with transits and a tape, and then draw the plan to scale. One or two attempted blue-prints, and another photographed his plan. Others verified their results with the specifications made from the deed by a surveyor. The following week the vertical plane of their instruments was used and with it they measured the angle of elevation of some building or pole, and then, by drawing to scale, estimated the height.

The next week we made alum and salt crystals. Alum crystallizes in octahedrons and salt in cubes. I gave no information as to what forms they would get, so when one of the boys came in bursting with the information that his crystals were just the shape of the candy boxes we had made, there was real enthusiasm that old Plato was right—God did eternally geometrize.

Closely following this came a very simple study of crystallography and the geometric forms of crystals. Some wrote essays on the various forms which occur among well-known minerals, others drew pictures of the minerals (we found several fascinating books in the library), others reported visits to the exhibits of mineralogical collections in the Agassiz Museum, telling about the geometry of the crystals, and still others brought in collections of their fathers or mothers or aunts or uncles or friends.

Very little class time is taken for these things, but the group around the bulletin boards or the display table was evidence enough that the geometry of nature was making an impression.

For the next week's work, geometric designs as illustrated in games was our problem. Each pupil drew accurately and to scale where possible (construction lines such as for perpendicu-

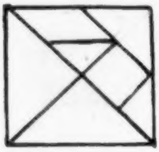
lars and bisectors to be left on finished work so as to show accurate work) the plan of at least three games. When one stops to consider, there are of course very few games which do not have a geometrical "set-up." It was astonishing even to some of the boys that a baseball diamond was a square! More than thirty different games were outlined, including football, basketball, track, swimming pool, steeple chase, hop-scotch, checkers, parchesi, and all sorts of other indoor and outdoor games. Those who preferred, drew house plans according to scale, the idea of this week's work being "scâle" drawing.

Further familiarity with polygons came the next week in the form of an old Chinese puzzle known as tangrams. In this puzzle a square is divided into four triangles, a parallelogram, and a square by a diagonal and four other lines. These other four lines are made by bisecting various lines or line segments and the proof of the parallelogram and the square makes an interesting exercise. After the square has been cut along the constructed lines an interesting puzzle is made. If the pieces are put together in certain ways, very clever pictures may be made with them, and the arrangement of these pieces to form designs and pictures is by no means an easy stunt, and the ways that the pieces can be put together gives a new familiarity with shape. The *Boston Globe*, in 1923, gave a series of pictures which could be made with the tangrams so that we had these for an incentive and start. The results were pleasing, and added, I think, considerable interest to the course.

While discussing tangrams we learned that there were a great number of such or similar puzzles on the market which were usually accompanied by booklets giving a number of designs, usually geometric, which could be made with them. For the next week we made a collection of such designs, and because it was near Christmas-time, found nearly a dozen such puzzles for sale in the department stores. We then had for our problem the construction of new and different puzzles of the same kind. Each pupil was to divide a square into triangles, etc., with a few simply constructed lines, and then with these pieces to make other designs and pictures.

It also happened that the *Boston Herald* in its Sunday supplement of that week had in its puzzle section a problem called the Glazier Puzzle. It consisted of five equal right triangles and

five equal trapezoids. The problem read "A glazier was ordered to make a square window with the scraps of glass shown. He succeeded. Are you as clever as he?" Enough pupils had the



paper or could trace the puzzle so that this was our next week's problem. In connection with this we had various other problems from newspapers which members of the class brought in, and these were solved by various members.

A great deal of geometry exercise work in proofs was done with these also, *e.g.*, proving that bisecting certain lines gave parallelograms, squares, etc., and the fact that we had something a little more tangible than a book-exercise added zest to the class work.

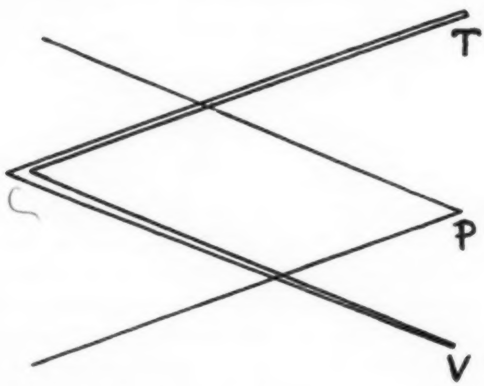
For several weeks after this the extra work was the construction of large figures of the theorems of geometry. These were drawn on large sheets of drawing paper so that they could easily be seen across the room. They not only gave practice in drawing larger accurate figures, but also afforded a quick review of the work without time wasted in class drawing the figures on the blackboard.

Magic squares, magic circles, designs on oriental rugs, primitive art, more complicated architectural designs, the geometric patterns spilled all over the sky at night, were other projects which claimed our attention other weeks.

At the time when we were studying locus problems in our class work, models were made of various locus theorems or exercises, merely to get into tangible form some of the work here. Embroidery hoops made the circles, fine wire or thread made the lines and showed the locus. Figures such as the locus of the vertex of a right triangle having a fixed hypotenuse, the locus of the midpoints of equal chords of a circle, the locus of the midpoints of chords drawn through a point of the circle (or within), the locus of the vertex of a triangle having a fixed base and a given vertex angle, etc., were constructed.

The next week we had begun the study of proportions and

similar triangles so that our next venture was with a pantograph. This is an instrument for making copies of a picture or a map to any scale required. In the most usual form, which we con-



structed, rods are hinged at each joint so that they can turn easily and the lengths are arranged so that $CT = CV$, $AP = AV$, $CB = AP$, and $CA = BP$.

The end V is fixed and a tracing point is placed at T, and a pencil at P. Then as T traces the outline of a picture, P reproduces it. If the picture is to be enlarged, the tracing point is placed at P and the pencil at T. All sorts of pictures were enlarged with the instruments which they made, from Andy Gump to the Campbell Kid; and the instrument itself gave us many class exercises.

Another field which we have touched on lightly is the field of dynamic symmetry. It seems to be the general opinion that it gives for those who wish it a skeleton outline on which to build a design or picture which is new and which presents many possibilities, *e.g.*, artists of considerable note and fame have used and applied its principles with most satisfactory results. It is enough for us in geometry to know that since the subject has gained recognition everywhere and since the groundwork is geometrical, there is an opportunity here to show how geometry may be used in art. We copied some designs made in this way, did several of the constructions in Hambidge's Dynamic Symmetry, measured rugs, pictures, fireplaces, rooms, etc., to find root rectangles—in other words merely became acquainted with what was meant by Dynamic Symmetry.

OBJECTIVES IN TEACHING INTERMEDIATE ALGEBRA

By PROFESSOR W. D. REEVE,
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In THE MATHEMATICS TEACHER for November, 1925, I contributed an article on "Objectives in the Teaching of Mathematics." A large part of the discussion was devoted to the objectives to be attained in teaching elementary algebra. I have had so many requests for reprints of the above article and so many comments as to its helpfulness to classroom teachers that I venture at this time to give a list of objectives in intermediate algebra.

As was the case in the list referred to above, the following list has been prepared with the help of a large number of Teachers College students who are experienced teachers of mathematics throughout the country. Moreover, the order of topics is not particularly important and no attempt has been made to exhaust all the possibilities. If the objectives suggested serve as a basis for clarifying our views with respect to the aims in teaching the subject, we shall feel that our efforts have been worth while.

Objectives in Teaching Functions

To develop an understanding of

1. Function—algebraic and transcendental.
2. Variable—dependent and independent.
3. Constant.
4. Functional notation of an elementary kind.
5. How functions are evaluated.
6. How simple functions are determined when several values of x and $f(x)$ are known.
7. The different classes of algebraic functions, their type forms, and the method of drawing their graphs.
8. Variation of functions, including maximum and minimum values.
9. How to interpret the graph of a function.

10. How to find maximum and minimum values graphically and algebraically for certain elementary functions.
11. How functions are used in the fields of science, economics, and business.

This includes a review and extension of the objectives in the teaching of formulas in ninth grade algebra.

To develop the ability to solve

1. Formal problems involving functions.
2. Applied problems involving functions.

Objectives in Teaching Factoring

To develop the ability to

1. Factor the following types:

$$ax + bx + cx = x(a + b + c). \tag{1}$$
$$a^2 + 2ab + b^2 = (a + b)^2. \tag{2}$$
$$a^2 - 2ab + b^2 = (a - b)^2. \tag{3}$$
$$a^2 - b^2 = (a + b)(a - b). \tag{4}$$
$$x^2 + (a + b)x + ab = (x + a)(x + b). \tag{5}$$
$$acx^2 + (ad + bc)x + bd = (ax + b)(cx + d). \tag{6}$$
$$ax + ay + bx + by = (a + b)(x + y). \tag{7}$$
$$a^3 + 3a^2b + 3ab^2 + b^3 = (a + b)^3. \tag{8}$$
$$a^3 - 3a^2b + 3ab^2 - b^3 = (a - b)^3. \tag{9}$$
$$a^3 + b^3 = (a + b)(a^2 - ab + b^2). \tag{10}$$
$$a^3 - b^3 = (a - b)(a^2 + ab + b^2). \tag{11}$$
$$a^4 + a^2b^2 + b^4 = (a^2 + b^2 + ab)(a^2 + b^2 - ab). \tag{12}$$

2. Use the factor theorem.
3. Use the remainder theorem.
4. Use synthetic division.

Objectives in Teaching Ratio, Variation, and Proportion

In addition to the objectives given in THE MATHEMATICS TEACHER, November, 1925, page 398, we may have the following:

To develop an understanding of

1. Ratio as a multiplier.
2. Joint variation.
3. The relation of variation to proportion.
4. The fundamental laws of proportion.

To develop the ability to

1. Solve formal problems in variation and proportion.
2. Solve applied problems in proportion.
3. Apply the laws of proportion to problems in physics, to changing recipes, and to the adjustment of patterns.

Objectives in Teaching Simultaneous Linear Equations

To review and extend or to develop the ability to

1. Solve linear equations in two unknowns:
   a. With integral coefficients.
   b. With fractional coefficients, both common and decimal.
   c. With literal coefficients.
2. Solve linear equations in two unknowns by means of determinants.

This presupposes the ability to expand a determinant of the second order.

3. Solve linear equations in three unknowns:
   a. By the ordinary method of elimination.
   b. By means of determinants.

This presupposes the ability to expand a determinant of the third order.

4. Check all results.
5. Solve applied problems involving two or three unknowns. Real problems in three unknowns are difficult to find, but there are a few.
6. See that there must be as many equations as there are unknowns.
7. To solve for the constants in equations of the type $ax + by = c$.

Objectives in Teaching Quadratic Equations in One Unknown

To review and extend the abilities outlined in the list of objectives for elementary algebra and

To develop the ability to

1. Solve higher-degree equations which are in quadratic form.
2. Solve literal quadratic equations.
3. Find the roots of the incomplete quadratic equation which results from the general quadratic $ax^2 + bx + c = 0$, when, for example, $b = 0$.

4. Understand the relation of the roots to the coefficients of the general quadratic $ax^2 + bx + c = 0$.
5. Derive formulas for the sum and product of the roots of a quadratic equation.
6. Determine the nature of the roots of a quadratic equation by using the discriminant $\sqrt{b^2 - 4ac}$.
7. Write a quadratic equation that shall have two given roots.

Objectives in Teaching Simultaneous Quadratic Equations

To develop the ability:

1. To solve a system containing one linear and one quadratic equation.
2. To solve a system of the type $\begin{cases} a_1x^2 + b_1y^2 = c_1, \\ a_2x^2 + b_2y^2 = c_2. \end{cases}$
3. To solve a system of the type $\begin{cases} ax^2 + by^2 = c, \\ xy = k. \end{cases}$
4. To pair the roots of such systems properly.
5. To check all solutions of such types.
6. To solve applied problems involving simultaneous quadratics.
7. To understand that one equation involving two unknowns is indeterminate and that to have definite values for n unknowns we must have n independent equations expressing relations between the unknowns.
8. To understand that, in general, a system involving two second-degree equations cannot be solved by elementary algebra.

Objectives in Teaching Higher Equations

To develop the following abilities:

1. To solve certain cubic equations by factoring.
2. To form a cubic equation containing given roots.
3. To find, if possible, the roots of a higher-degree equation by substituting a for x or by dividing by $x - a$.
4. To solve a system of the following types:

   a. $\begin{cases} x^3 + y^3 = a, \\ x \pm y = b. \end{cases}$

   b. $\begin{cases} x^4 - y^4 = a, \\ x^2 \pm y^2 = b. \end{cases}$
5. To know that in higher algebra it is proved that every equation of the nth degree has exactly n roots and no more.

6. To show by solving the equation $x^3 = 1$ that there are three cube roots of unity.
7. To know that, by higher algebra, cubic and biquadratic equations can be solved by formula.
8. To know that, in general, equations of the fifth and higher degrees cannot be solved by algebra.

? $x - \sqrt{-1} = 0$

9. To know that imaginary roots always enter in pairs.

Objectives in Teaching Graphs

To develop the following abilities:

1. To solve graphically a system of one quadratic and one linear equation.
2. To solve simple simultaneous quadratic equations by graph.
3. To construct and interpret the graph of any quadratic equation of the type $y = ax^2 + bx + c$.
4. To graph the equations $x^2 + y^2 = r^2$, $ax^2 + by^2 = c$, $x^2 - y^2 = a^2$, $xy = k$ and $y^2 = ax$, and to know the names of the curves so drawn.
5. To understand that the form of a graph is determined by the constants in the equation.
6. To be able to determine, in certain simple cases, the character of a graph by looking at its equation.
7. To be able to approximate from its graph the roots of an equation of higher degree than the second.
8. To find from its graph the maximum and minimum values of a function.
9. To understand, by the study of the graph, how many pairs of roots to expect in a quadratic system.
10. To recognize from its graph the general characteristics of a cubic equation in one variable.

Objectives in Teaching the Binomial Theorem

To develop the following abilities:

1. To find by actual multiplication the first six powers of $a + b$.
2. To understand how to write the letters and exponents of powers by inspection.
3. To be able to make Pascal's triangle and to use it in determining coefficients.
4. To know Newton's method for determining coefficients.

5. To understand how to use the binomial formula in making expansions.
6. To be able to write the first few terms of the expansion $(a+b)^n$ correctly.
7. To find the kth term of the expansion of a binomial.
8. To prove by mathematical induction the binomial theorem for positive integral exponents.
9. To use the binomial theorem in problems involving compound interest.
10. To understand that the binomial formula is the mathematical basis of compound interest.

Objectives in Teaching Arithmetic Progressions

To develop the following abilities:

1. To understand the nature of an arithmetic progression.
2. To be able to continue such progressions as are illustrated by 1, 4, 7, 10, 13, $\cdots$ and 11, 6, 1, $\cdots$.
3. To derive the usual formulas $l = a + (n-1)d$ and $S = \frac{1}{2}n(a+l)$.
4. To derive other formulas from those in 3.
5. To use the formulas of (3) and (4) in solving problems.
6. To insert one or more arithmetic means in a series of not over six terms.

Objectives in Teaching Geometric Progressions

To develop the following abilities:

1. To understand the nature of a geometric progression.
2. To derive the formulas

$$l = ar^{n-1} \qquad \text{and} \qquad S = \frac{a(1-r^n)}{1-r}.$$

3. From the formulas of (2) to derive others.
4. To use the formulas of (2) and (3) in the solution of problems.
5. To be able to continue such progressions as are illustrated by 3, 12, 48, $\cdots$ and $\frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \cdots$.
6. To derive and use the following formula:

$$S = \frac{a}{1-r}.$$

7. To know that an infinite decreasing geometric series illustrates the idea of a limit.
8. To understand the formula for the sum of an infinite geometric series in relation to repeating decimals.
9. To understand the nature of an infinite geometric series.
10. To insert one or more geometric means in a series of not over five terms.
11. To derive and use a formula for the amount, at the end of any number of years, of a sum of money deposited annually, interest compounded annually.

Objectives in Teaching Exponents and Radicals

To develop the following abilities:

1. To know the definitions of negative, zero, and fractional exponents.
2. To know and to use the law $a^m \times a^n = a^{m+n}$.
3. To know and to use the law $\frac{a^m}{a^n} = a^{m-n}$.
4. To know and to use the law $(a^m)^n = a^{mn}$.
5. To know and to use the law $(abc)^m = a^m b^m c^m$.
6. To know and to use the law $\left(\frac{a}{b}\right)^m = \frac{a^m}{b^m}$.
7. To prove the law $a^m \times a^n = a^{m+n}$ for positive integral exponents.
8. To prove the law $(a^m/a^n) = a^{m-n}$, m being greater than n, for positive integral exponents.
9. To prove the law $(a^m)^n = a^{mn}$ for positive integral exponents.
10. To perform the fundamental operations with expressions containing positive, zero, and negative exponents.
11. To understand what is meant by the following:
    a. Surd.
    b. Power.
    c. Root.
    d. Radical.
    e. Rational number.
    f. Extraneous root.
    g. Index.
    h. Mixed surd.
    i. Entire surd.
    j. Quadratic surd.
    k. Similar radicals.
    l. Radical sign.
    m. Irrational number.
    n. Radical equation.

12. To simplify radicals of the following types:

   a. $\sqrt[n]{a^m} = \sqrt[kn]{a^{km}}$.

   b. $\sqrt[n]{a^n b} = a\sqrt[n]{b}$.

   c. $(\sqrt[n]{a})^n = \sqrt[n]{a^n} = a$.

   d. $\sqrt[n]{a^m} \cdot \sqrt[q]{b^p} = \sqrt[nq]{a^{mq}b^{np}}$.

   e. $\dfrac{\sqrt[n]{a^m}}{\sqrt[q]{a^p}} = \dfrac{\sqrt[nq]{a^{mq}}}{\sqrt[nq]{a^{np}}} = \sqrt[nq]{a^{mq-np}}$.

   f. $\sqrt[n]{\dfrac{a}{b}} = \dfrac{\sqrt[n]{a}}{\sqrt[n]{b}} = \dfrac{1}{b}\sqrt[n]{ab^{n-1}}$.

   g. $\dfrac{a}{\sqrt{b} \pm \sqrt{c}} = \dfrac{a(\sqrt{b} \mp \sqrt{c})}{b \mp c}$.

13. To reduce radicals to other simpler forms.
14. To change the index of a radical.
15. To introduce a coefficient under the radical sign.

This includes introducing a denominator under the radical sign.

16. To use the fundamental operations with expressions involving radicals.
17. To transform an expression which contains fractional exponents to one which contains radicals and to reverse the transformation.
18. To transform expressions containing negative exponents to those containing positive ones, and vice versa.
19. To evaluate the radical $5a\sqrt{7x^2 - ay}$ when $a = 3.7$, $x = 4$, $y = 2.8$, and to make similar evaluations.
20. To find the square root of such expressions as $\sqrt{13 - 3\sqrt{7}}$.
21. To find the square root of a polynomial.
22. To find the square root of a number to two decimal places.
23. To find roots by tables.
24. To use in finding roots the following approximation formulas:

$$\sqrt{a^2 + r} = a + \frac{r}{2a} \quad \text{and} \quad \sqrt[3]{a^3 + r} = a + \frac{r}{3a}.$$

a^2

25. To manipulate formulas containing radicals.
26. To solve simple radical equations.

27. To solve radical equations that are no more difficult than $\sqrt{3 - 2x} - x = 30$ and to check such equations for the purpose of discovering extraneous roots and verifying any that are not extraneous.
28. To understand what is meant by finding a root to a certain number of significant figures.
29. To solve applied problems involving radicals.

Objectives in Teaching Logarithms

To develop the following abilities:

1. To understand and to appreciate how the work in logarithms grew out of a need for saving labor by shortening computation.
2. To know something of the history of the development of logarithms.
3. To appreciate the value of logarithms as a device for saving time and labor.
4. To make a brief table of logarithms by means of exponents.
5. To discover, by means of a table of the powers of 2, how to solve such exercises as 16×64, $2048 \div 32$, and $84\sqrt[3]{4096}$.
6. To make tables of the powers of other numbers than 2, and to solve problems by using these tables.
7. To know that the logarithm of a number is the exponent of a certain number—in practical cases, 10.
8. To know that 10 is the base of the common system of logarithms.
9. To know how certain laws of exponents are related to logarithms. For example, how $a^m \times a^n = a^{m+n}$ is related to $\log ab = \log a + \log b$.
10. To prove and use the law $\log ab = \log a + \log b$.
11. To prove and use the law $\log (a/b) = \log a - \log b$.
12. To prove and use the law $\log a^b = b \log a$.
13. To prove and use the law $\log \sqrt[b]{a} = (1/b) \log a$.
14. To compute with logarithms.
15. To use a logarithmic table.
16. To find the logarithm of a number in a four-place table.
17. To find the antilogarithm of a number in a four-place table.
18. To make necessary interpolations in computation.
19. To use the rule for finding the characteristic of the logarithm of a number, but not until the reason for the rule is clear.

20. To understand and to use cologarithms.
21. To use the expressions "integral part of the logarithm" and "decimal part of a logarithm" interchangeably with "characteristic" and "mantissa," respectively.
22. To apply logarithms in the work in trigonometry.
23. To evaluate certain formulas, getting results correct to three or four significant figures, by using logarithms.
26. To solve certain problems involving higher powers and roots by means of logarithms.
27. To solve interest problems by using logarithms.
28. To use logarithms in solving simple exponential equations.
29. To transform an exponential equation into a logarithmic equation and inversely.
30. To graph the equation $x = 10^y$.
31. To interpret the graph of the equation $x = 10^y$.

Objectives in the History of Algebra

To acquire the knowledge of the subject to this extent:

1. To know a few of the distinctive features of about four periods, as follows:
   a. The period of ancient algebra with almost no algebraic symbolism, with one or two great names and their contributions.
   b. The period of the development of modern symbolism, say from 1550 to 1650, with two great names.
   c. The period of the extension of our number systems to include positive and negative and imaginary numbers, with one or two great names.
2. To know a few interesting facts relating to men like Diophantus, Tartaglia, Cardan, Descartes, Fermat, and Newton, and to be able to tell about when and where they lived.[1]
3. To appreciate the progress of algebra.
4. To appreciate that mathematics is a growing subject by a consideration of historical material.

Objectives in Teaching Complex Numbers

To develop the following abilities:

1. To interpret the graph of an imaginary number like $\sqrt{-1}$.

[1] See D. E. Smith, *History of Mathematics*, Vols. I and II. Ginn, Boston, 1923 and 1925.

2. To graph and work with complex numbers in the usual form $a + b\sqrt{-1}$.
3. To know the definition of the imaginary unit.
4. To distinguish between a real number and the so-called imaginary number.
5. To be able to perform the fundamental operations with simple complex numbers.
6. To be able to solve certain of the simpler equations containing complex and imaginary numbers.
7. To reduce the sum, difference, product, and quotient of two complex numbers to the form $a + bi$.

THE PROBLEM OF ALGEBRA INSTRUCTION

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With the introduction of the Junior High School movement, mathematical instruction in our schools is at present in a period of transition. In the old system, arithmetic was presumed to be completed in the eighth grade and algebra started in what was then termed the first year of high school. With the newer idea, a course in general mathematics is given during the first two years of the Junior High School followed by algebra in the last year and then the higher mathematics work of the senior high school.

By the end of the sixth grade it is assumed that the pupil should be able to perform with accuracy, and a fair degree of speed, the fundamental operations with integers, common and decimal fractions. Then there follows for two years a "composite," "correlated," "unified" or "general" course in mathematics, the purpose of which is to enable the pupil to gain a broad view of the whole field of elementary mathematics early in his high school course. The advocates of this new method of organization base their claims on the obvious and important interrelations between arithmetic, algebra and geometry. It is advocated that this new correlation offers a more effective psychological and pedagogical approach to the study of mathematics than by the older rigid division into "subjects" such as arithmetic, algebra, geometry and trigonometry. This is undoubtedly true unless the child has had superficial training up to and including the sixth grade, in which case general mathematics, and in fact all the subsequent mathematics courses, will prove burdensome to the student for lack of a knowledge of the fundamental processes. Not infrequently students have passed into the algebra classes with a knowledge of fractions so superficial that it has been necessary to reteach them in arithmetic before passing to the various processes with fractions in algebra. But this is no argument against general mathematics. It simply

shows the need for a thorough training in the fundamentals before the general mathematics course, and then frequent drills and reviews of past work. This necessary review of basic principles should enter as a by-product and in that way avoid dulling the interest in new topics.

The Values of Mathematics

Some of our eminent educators are pressing the question: "Why should pupils study mathematics higher than practical arithmetic, especially if they do not intend to follow professions demanding a knowledge of mathematics?" The answer is simple:

Mathematics exemplifies most typically, clearly and simply certain modes of thought which are of the utmost importance to everyone. The task is to train the students in correct and useful habits of cogent thinking—to encourage them to face problems sanely and rationally and use their brains. It is a requisite for success in any occupation to be able to grasp a situation, to seize the facts and to perceive correctly the state of affairs. Mathematics is especially adapted to the beginning of this practice. No other subject in the school curriculum is its rival in this regard. Every branch of mathematics from the first grade arithmetic to the end of calculus carries out this idea.

A further argument for the extensive study of mathematics is the intimacy with which it is connected with everyday life. Wherever one may turn in these days of iron, steam and electricity, mathematics stands out as the pioneer and checks and guarantees the results. It has been said that were the backbone of mathematics removed, our material civilization would inevitably collapse. It is true that to the large majority of people its importance, although great, is indirect and the average citizen has little utilitary need for mathematics beyond arithmetic. But this could be said of any subject. Only the elements of English are in constant use. When is a person required to name the parts of a flower, give the date of the Battle of Hastings, decline a Latin verb or state the laws of gravity? Of all the subjects taught in school it is difficult to find a single one of which use is made in the occupations of life by substantially all the pupils. But that is no argument for their removal from the course of study.

This insistence for a utilitary criterion for each subject will spell the doom for culture unless there is a reaction. The high school caters too much to what the machinist, the carpenter, the hardwareman and the dressmaker think the children ought to be taught rather than attempting to liberate the mind and train for honest citizenship. A far greater need is for minds developed to think closely, resist delusion, discriminate and do some original thinking, than for skillful fingers and unresisting minds.

With our complex civilization it is becoming more and more difficult for a boy or girl to definitely decide at an early age just what occupation he or she desires to follow in after life. A subject is therefore valuable as a preparation for the contingency that the child in the future may take up as an occupation. This is especially true of mathematics, for there is a large and growing number of occupations which require a knowledge thereof.

The National Committee on Mathematical Requirements in their report "The Reorganization of Mathematics in Secondary Education" state especially in regard to algebra: "Of about equal importance to every educated person is the understanding of the language of algebra and the ability to use this language intelligently and readily in the expression of such simple quantitative relations as occur in every-day life, and in the normal reading of the educated person." It offers the only means of solving many of the most important problems connected with engineering of all kinds, architecture, navigation, geology, physiology, physics, chemistry and many others. It is literally true that no mechanic can read intelligently his trade journal, a technical book or a scientific article in an encyclopedia without a knowledge of the language of algebra.

"Appreciation of the significance of formulas and ability to political problems will also be recognized as one of the necessary equations must nowadays be included among the minimum requirements of any person of universal education.

"The ability to understand and interpret correctly graphical representations of various kinds such as nowadays abound in popular discussions of current scientific, social, industrial and political problems will also be recognized as one of the necessary aims in the education of every individual. This applies to the representation of Statistical data, which is becoming increasingly important in the consideration of our daily problems, as

well as to the representation and understanding of various sorts of dependence of one variable quantity upon another."

The transfer and disciplinary value of mathematics cannot be overlooked. This problem has for a number of years been a controversial one, but now psychologists are pretty well agreed that the effects of training do transfer from one field of learning to another. The amount of transfer of course depends upon a number of conditions, but training in connection with certain attitudes, ideals and ideas is almost universally admitted by psychologists. It may be said very truly that the general mental discipline derived from mathematics is warrantable grounds to include such training among the valuable aims or purposes of algebra.

Preconceived Notions about Algebra

The notion has become prevalent that algebra is difficult and many students have entered algebra classes in fear and trembling. This conception of its difficulty has been handed down from parents or from brothers and sisters who have previously failed the subject. And unless the algebra teacher be able to dispel such an erroneous belief, students holding it in most cases will fail.

There are those who argue that only those possessing an exceptionally high intelligence quotient should study algebra. Mathematics instruction should not be denied any child, but a boy or girl who has a very low intelligence and who has barely passed arithmetic should not pursue the same work in algebra as one who is high in the scale. Attempts have been made to remedy this condition by grouping the students. But in many schools that is as far as it goes, for they are required to cover exactly the same work in the same time, and at the end of the year all groups take the same examination. The value of student grouping is yet to be discovered under such conditions, yet in New York State, where grouping is in vogue, identically the same regent examination goes to each student. The sensible way to take advantage of the grouping is to use different methods of presenting the various topics and drill more on some than on others. Even this becomes impossible in systems where departmental tests are given every four weeks to all the classes, presuming that each group has covered the same amount of work

in the same time. If there is going to be student grouping, then each group ought to have a list of topics to be covered and an examination appropriate to the group.

Frequently parents argue that their child does not like algebra and for that reason ought not to be compelled to pursue that study. Schoolwork, like life, consists in doing many things which are not exactly pleasurable and if school is a training for life, this situation needs to be met. Thorndike, after extensive studies, found that a 0.9 correlation exists between abilities and interests (0.4 is considerable and 0.7 is high) and except under very exceptional circumstances students get relatively more out of subjects that their special abilities enable them to work most effectively. But even so, that is no argument. Such students will not lead the class, neither should they be allowed to dangle along on the end.

First Impression of Algebra

There is no moment more critical in the classroom than the first time the class assembles. While visiting a class at such a time, the teacher in introducing algebra told the students of the difficulty of the "new" subject and the need for prolonged study. Their enthusiasm was immediately squelched. A tide of discouragement swept over the room and the failure of many a student dated from that day.

Another skillful teacher held the class spellbound in telling them something of the history of mathematics and working a few practical algebra problems on the board. The value of an interesting introduction cannot be overestimated, for coming from the grades with no real clear plan for the future and encountering a new set of subjects, the very names of which are often strange, it is no wonder that students become confused for a time and are easily discouraged. Some pupils have a natural antipathy towards education and regard the whole process as a set of tasks intended in some undefined way to militate against their own desires. Still others may be anxious to find out how education in general and algebra in particular will prove of value to them. The essential thing is to make the pupils at home—to arouse enthusiasm for the subject and make them look forward to their work with pleasure. For efficient work, interest must be kept at high pitch from the beginning to the end.

Organization of Subject Matter

The assertion has been freely made that the vital point in actual mathematical instruction is in the ability to pass an examination at the end of the course. The real value, however, hinges about the child's needs and capacities. This does not mean that less value is attached to the strict logic of algebra or that it is to be made less clear to the pupils. On the contrary it may well be claimed that by minimizing the merely technical manipulation, by omitting work whose only merit is its complexity and including as much practical drill as possible the logical aspects of the subject may not suffer and its interest value be greatly increased.

Exercises which have no value other than "drill for drill's sake" are of doubtful utility. A simple example will test out a pupil's knowledge, and the same kind will give him practice, without introducing the discouraging elements which come with a long, complicated and meaningless one. The subject will suffer no serious loss from the omission of unduly complicated processes that are not likely ever to be found either in future mathematics or in the physical sciences.

The algebra of the Junior High School should be the outcome of concrete problems as far as possible. Many may be given requiring merely that a relation given in words be stated in the form of an equation. They may begin with the very simplest, and gradually increase in complexity to the most complicated statements involved in any problem to be given and a knowledge of algebraic manipulation to solve them. Algebra should not be arithmetic made difficult, but rather a more versatile and powerful tool for mathematical work.

The equation which is the core of the course is a method for solving problems and not finding decimal values for unknown terms. The formulas very easily can be made to apply to practical computation taken from the shop, trade journals and engineering work. Interest in them may be aroused by showing the advantage of algebraic over arithmetical methods in problems of percentage, discount and bank interest. The rules, principles and processes then are rationalized and the operations of solving them, growing out of a created need, are introduced as desired. In this connection Breslich says: "Algebra taught as

an organized science is to a pupil of this age nothing but a mechanical juggling of symbols, a wearisome iteration of meaningless manipulation, a waste of time which stunts his intellectual growth. But give a pupil a wide experience with these fundamentals and it will make possible a gradual and easy approach to the parts of the subject that are later taught in the high school.'' More than that: with progress in the work there should come an increased interest in and desire for mathematical work.

The entire course in algebra must be so organized as to place special emphasis on the development of ability to grasp and utilize ideas, processes and principles rather than acquire a mere facility or skill in manipulation. One of the obstacles to intelligent progress is the excessive emphasis now commonly placed on manipulation beyond what is needed as drill. In order to realize this end there are three principles of criteria for exclusion: (1) All items which are not themselves directly used in practical situations or which are not reasonably necessary to the intelligent mastery should be eliminated. (2) All complicated instances of useful topics or application should be simplified so that they will perform their mission without undue difficulty. (3) All discussions likewise which do not have an intelligent use should be excluded.

The Need for Sympathy and Optimism

The keynote of successful teaching is sympathy. This does not imply a lessening of the assigned work or an inflation of the students' marks, but rather a patience and willingness on the part of the teacher to do all that can be done for the student. A sympathetic teacher, realizing the problems of the child, will do all that is possible to meet discouragement with bright optimism. A word of praise and ''sure you can'' will do wonders. There will also be a word of encouragement for a successful piece of work; a hearty laugh at a comical situation and a good time for everyone in the class. Work and a good time can be made to go hand in hand by an artful teacher.

A teacher must create inspiration. Any device or work of commendation which will arouse some weak or sluggish mind to action, and take to a new grasp should be freely taken into the schoolroom. Schoolwork is life; and when life is void of inspira-

tion, existence becomes dull and monotonous. The life of many a boy and girl has been altered by the influence of a sympathetic, inspiring and venerable teacher.

Homework and the Development of New Work

The manner in which the homework assignment is made in algebra is more important than in any other subject. To assign the examples in exercise X for homework is absolutely meaningless. The best method is to work a representative example on the board, being careful to explain the necessity for each step, and willing to answer questions relative to the work, while the example is actually being worked. If the questions are reserved for the end of the example, the students have either forgotten them or in the activity of trying to remember them failed to grasp the work being done. Then the teacher should allow the class to select another example to be worked. Sometimes it is best to send a student to the board to work the example while at others it is better for the teacher to act as clerk and work as directed by students which are designated to contribute to the solution. Every mark put upon the board should be first supplied by the student reciting and accepted by all as correct. Thus each member becomes personally interested in the operations and alert to give directions or detect errors. The class is thus kept up to a high tension and intensive work may be accomplished. The boardwork becomes both a check and a key and the pupils feel that they have had a real part in its development.

In this way two typical examples have been worked out—one by the teacher and explained and the other by individuals in the class. When new types of examples are being introduced for the first time, while they are still on the board, a summary or method of procedure should be formulated, which will constitute a rule or direction for work. In solving simultaneous equations by the method of addition or subtraction for instance, the following summary serves as an illustration:

(1) Write so that the unknown quantities are on the left and the known quantities on the right of each equation. Be sure to change the sign of every term transposed.

(2) Multiply each equation by some number so that either the X's or Y's will be identical in both equations.

(3) Add or subtract these equations, using whatever method will eliminate one unknown.

(4) Solve the simple equation just found for the remaining unknown.

(5) Substitute this value in either of the original equations to secure a value for the other unknown.

(6) Check in the original equation by substituting the value found for the unknowns.

This method of formulating directions for procedure can be carried out for all the various processes in algebra. It is far superior to the old method of learning a rule and then blindly following it if possible. Thorndyke has found out that the inability of authors to formulate definitions and explanations in English comprehensible to the immature pupils has been a factor chiefly contributing to the large percentage of failure in algebra. By this method the student has actually seen the example worked, and then in his own words he has given directions for its solution. In this way algebra can be divided up into major units and each one related to previous work.

The Function of Memory

Shall these directions be memorized? That depends on just what is meant by memorizing. There is no subject that can be made more dull than mathematics when just rote memory is used. But there is a different kind of memory: that which is based on reasoning and remembering the sequence rather than the thing itself. It is this type which is valuable. The child must have a foundation on which to build, and these processes serve as such. They in some manner must be mastered and always ready for use. Memory plus drill works most effectively.

Drill and Homework

The drill which a teacher gives when accelerated by speed tests and similar schemes should be limited to those processes and to the degree of complexity required for a thorough understanding of principles and their probable application. Drill for drill's sake, if that be conceivable, is valueless. It must be conceived throughout as a means to an end, and not an end in itself. Within these limits, skill in algebraic manipulation is important, and drill in this subject should be extended far enough to enable

students to carry out the essential processes accurately and expeditiously. Students of the adolescent age require more drill than at any other period in their school life, and when a teacher loses sight of the necessity for drill, then the pedagogy of the teacher is superficial. The child's mind demands repetition and then more, until the very word drill becomes obnoxious to the ears of the teacher.

This drill work may be accomplished by means of homework and boardwork or their combination. The newer idea of supervised study has supplanted the old homework, the essential difference being that in one case the work is done (or copied) at home and in the other in the schoolroom under the supervision of the teacher, who should at all times be ready to offer judicious help.

After the work has been done on the board by the teacher and rules or methods formulated by the students, then the homework takes on the character of completing the work rather than preparing for the next day. One of the follies of algebra teaching is to set the pupil struggling with really new topics without this previous explanation by the teacher. The minds of children have not been developed sufficiently for much of that kind of work. The examples assigned may require thinking—problems may now be given for solution, but there should always have been enough similar antecedent work to furnish pupils a clue to prevent working in the dark.

Throughout the entire study work, whether it be done at home or during supervised study, the teacher should be carefully studying and diagnosing each pupil. The silent study period gives the teacher the best opportunity for this individual personnel, but a good idea can be obtained from the papers handed in—their accuracy, neatness, degree of completion and recurrence of faults. Often a pupil, who would ordinarily have failed, has found himself through a little attention and study on the teacher's part. The elimination of pupils from the list of failures should become the predominating effort rather than the elimination of pupils from class and school.

A very excellent way to take care of individual differences and encourage pupils is to assign a minimum and maximum assignment, allowing extra credit for work done in the maximum assignment. This maximum assignment will take care of the

brighter pupils—those who are able to do more than the average pupil and who should have an incentive to do advanced work. Care must be taken to so plan that it will not discourage, but incite the pupil's desire to complete it. This likewise can be so fashioned as to constitute an excellent review of previous work. The pupil then becomes conscious of something positive having been accomplished.

The major purpose of recitations, drill, boardwork and homework is to set the pupil thinking. The superior teacher is the one who gets the pupils to do the thinking and talking and any devices such as mathematical games, reports, clubs or intercurricular activities which will arouse them to such activities are commendable.

The Algebra Teacher

A discussion of algebra would not be complete without reference to the teacher, for it is pretty generally conceded that algebra is the most difficult of all subjects to teach, and its teaching should not be undertaken without a thorough preparation not only in subject matter but also in the art of presenting it.

It is a lamentable fact that far too many are using teaching to bridge the gap between maidenhood and marriage rather than being driven by an inner urge. They are not the servants of an idea or of a passion of the soul. Most of them could easily have been something else; could have taught something else. Speaking of teachers in general Lewisohn says: "They went into teaching either because they had a pleasant taste for learning and no particular taste for anything else; or because they were timid and of a retiring nature and didn't like the rough and tumble of the business world. Or because—in an appalling number of cases—they simply drifted into academic life. Thus there is among them little intensity or power, little courage or independence, much pinchbeck dignity and lust for administrative twaddle." This may seem hard on teachers, but nevertheless it is true in far too many cases. The best mathematics teachers are those who positively could not have been successfully placed anywhere else—who have an inborn desire for mathematics and who can apply the electric torch of personality. Until a boy or girl comes under such a teacher he has not known the glory of mathematics.

There is a tendency on the part of principals to assign several different subjects to a teacher. The result is that neither subject is properly taught. A teacher needs specialization and a well-rounded experience in the subject taught. This cannot be obtained when each year a teacher is switched from one subject to another. Angular, strawy minds, filled with just the minimum essentials for the subjects, and a narrow, technical sort of teaching without richness or savor are the inevitable results. These "nomadic subject" teachers often think they are teaching when they are only modestly sitting behind a desk, pencil in hand, and a serious look on their faces. In reality they are engaged in the tiresome occupation of killing time. They likewise think they are "covering the work" when they are only asking book questions without developing the reasoning faculties.

During the last few years our entire mathematical course has gone through a transition in regard to subject matter and teaching methods. But the desired results will never be achieved until there is a modification of teachers themselves, for after all the earnestness and personality of the teacher eclipse all else.

DIRECT CULTURAL MOTIVATION FOR DEMONSTRATIVE GEOMETRY

By P. STROUP,
Cleveland, Ohio

How early can the normal person be interested in the action and habits of his own mind? If it is at high school age or before, then we are missing a good road to his interest and attention by not calling his attention frequently to the simple mental actions that are common to all of us and to his own strengths and weaknesses. Who does not desire a strong and active brain? Would not boys and girls be attracted to mental improvement as they are enthused by physical exercise? No doubt many would follow the pernicious American habit of watching the other fellow take the exercise, but the appeal would inspire others by the visualization of a closer desirable objective. The object of education is self improvement and that no doubt is in the background of everyone's mind who is really seeking an education. But what I propose is more definite than that. It is the directing of the student's mind to his own mind to know how best to use it, not a general study but with lessons taken whenever they occur in the experience of the geometry class. There are many practical books on mind study that are helpful to teacher and student but even if used as a text in another course they would not take the place of the application of their principles to the daily experience in geometry or other class. In all my experience students have been found to be interested in anything definite along this line.

Students can be shown that part of the actions of their minds take place outside of their control. It is interesting to try to find out from a student why he has made a given reply. It quite amazes him to find out that his mind has played a little trick on him and to see that what he thought he was so sure of has not sufficient reason. A good example of this is when one angle of a triangle is 90° and he is asked what the other angles must be. Some in every class will have trouble seeing that they do not

have to be 45° each. Even after some of the other possible combinations have been shown them they will be caught on short notice by the same question. Another recurring example is the unconscious assumption by students that things that appear to be true are true. Many of our mental as well as our physical actions must be habitual so that the matter of improving our minds is largely a matter of refining our mental habits. It is well in passing to call their attention to the subconscious mind. Many people have had the experience of having their minds find the solution to a problem after they had worked hard at it for some time and then had slept or had turned their attention to some other matter. This can be held up as an ideal for the encouragement of continued attack on a hard problem. This suggests a resource of our minds that many are unaware of and few use. I found my students much interested in this possibility.

There is a common belief that a^2/b^2 equals a/b. This arises from a confusion of the operations possible with the equation and those possible with a fraction. The similarity of the terms used 'same to both' has caused a mental connection that must be broken. The pun is the lowest form of humor. The connection by similar sound is an elemental and much used habit but it must be checked strongly in logical thinking. There is a distinction between telling the student that he made a mistake and trying to find out how his mind made the mistake. In the former case what is to be done about it? In the latter case there is a trick of his mind that can be watched.

There are frequent instances of the habit of imitation carrying a student along unconsciously through a mere form when the results if looked at closely are absurd. A student having detected this habit of his mind would surely progress by being on his guard against it. I can remember very definitely when I thought I knew how to write because I could make a wiggley line across the paper which looked like writing to me.

The converse idea can be "sold" to the students as a useful and practical habit of mind for getting better acquainted with any subject and as a source of ideas. An example of this which appeals to most students is a variation of the recent dance craze, the Charleston. In this you start with the heels together and wiggle them as you progress. In the variation you start

with the toes together and wiggle them (collectively). Is this stretching the converse idea? Well, its elasticity needs to be shown to the students. I doubt that the inventor of the above-mentioned idea was a mathematician but I think I can follow his line of thought.

Paying attention is more than a matter of merely looking at the person who is speaking. In fact looking is not the most essential part. It is more a matter of asking questions of yourself about what is said, of connecting it up with what is already in your mind, of seeing what it suggests to you. Every such connection is a bit of buoyancy added to the idea to keep it from sinking into oblivion and to enable it to rise more easily when it has sunk below the surface of consciousness. This suggests that we as teachers often talk too fast. Paying attention is more than a matter of good intention on the part of the student. He can be helped in a practical way. It is not so much a matter of going in one ear and out the other. It goes in at both ears in the normal child but sinks in the bottomless sea of subconsciousness. Every serious question or suggestion should be welcomed in the class and the students frequently reminded that the rising of questions in their minds is the surest indication that they are really thinking.

Another reason why you cannot pay attention with a passive mind is illustrated by this instance. The class was shown two methods of finding the mean proportional between two lines by construction. A student who went to the board next day showed a thorough mixture of the two methods. He can be shown that his mind should have been active enough to have separated the operations of one method from those of the other. He let them drift in all in one pile. He probably thought he was paying attention and he probably looked as if he were. He is like a person who is neat enough to put his clothes in the closet but does not hang them where they belong. When he reaches in the closet, he gets something but seldom gets what he wants. Some minds do such work without conscious effort but it seems feasible that a mind lacking such a habit could foster its growth by giving it attention. Either we are merely teaching the facts of geometry or we are trying to accomplish this sort of thing. If it is the latter to any worth-while degree, then the process is not to be hurried and the call to speed up the course in mathematics cannot be heeded without detriment to our accomplishment.

Demands for attention can be based on the desire for self improvement. You can't learn tennis on a park bench. Mental improvement comes from mental exercise. If he believes that he could think if he tried, it is fair to suggest that geometry is a proving ground for straight thinking. If he can think straight he can do it in geometry and if he doesn't he can be shown just how far he is right and just what ideas are wrong.

In assisting a student in a recitation we seldom tell him just what we want him to say. We ask questions in an endeavor to show him that the required mental action is possible for his mind. If he had asked himself the same questions, he would not have needed help. Thinking can be helpfully defined as asking one's self questions. The question then becomes what questions to ask, but many useful suggestions can be given to that. What is given? What is to be proved? How can that sort of fact be proved? What is known of that kind of figure? In taking up these suggestions we can expect only the most gradual improvement on the part of those students to whom they are to be a real help. It is hard not to see the ridiculous in the gropings of a blinded person when we can see clearly. Would that we as teachers could as easily get the feelings of dull students as by blindfolding we can get the feelings of those whose sight is defective or lacking. What helps me most is to watch myself pursue those branches of higher mathematics which I did not take in college and for that matter those that I did. I am encouraged in finding that the suggestions that I make for students are helpful to me.

The importance of defining terms and of thinking of their definitions should be shown at every opportunity. When it is to be proved that a thing is half, the question of how that can be proved depends on the definition of half. The usual definition is that half is half. In studying law this rule for defining was given. Assign the thing to a class and then differentiate it from the rest of that class. That idea absorbed and applied will be a lifelong help to anyone who tries to think.

Students need to be impressed with the difference between the productive and reproductive functions of the mind. Although no manufacturing can be done without raw materials, there is a big distinction between a factory with good receiving facilities and a mere storehouse. A storehouse is a dull place.

The factory teems with interest and action. The man who accomplishes work that compels attention is he in whose mind the storage and manufacturing branches are both well developed and related. Efficient help for the memory should be given and the habit of using them encouraged. They should be logical rather than accidental whenever possible.

The first sentence of this article was a question and there should be a large question mark at the end. It is a question that is reasonable enough to deserve an answer. Psychologists tell us that students make most progress toward a given goal when they are conscious of what they are trying to attain and how they are expected to attain it. It seems feasible then to hold definitely before the student the direct objective of mind improvement. There is often the reservation in the student's mind that while he may not think clearly in geometry he does in his other affairs. He can be jarred from his complacency by showing him that the facts of geometry are dealt with by the same fundamental rules of common sense as any other facts of nature and either he does not know those rules or does not use them.

I do not believe that this adds complexity to an already complex subject. The word psychology should not be more than mentioned and the words of psychology should not be mentioned. Whatever added burden there is would be more than offset by the motivation it would furnish for the students who come to the subject with the impression and often the expression that geometry is to have little value to them.

As to the probability of interesting students in the actions of their own minds, Galton after extensive investigating reports "Many persons, especially the women and intelligent children, take pleasure in introspection, and strive their best to explain their mental processes."

NEWS NOTES

PROFESSOR WILFRED H. SHERK, of the University of Buffalo, is the new president of the Mathematics Section of the Middle States and Maryland Association of Teachers of Mathematics.

He wishes to receive communications from the secretaries of all of the local mathematics sections of the Middle States and Maryland Associations. Dr. Sherk is very anxious to get in touch with all of the sections and has requested that this announcement be published in THE MATHEMATICS TEACHER.

AT the January meeting of the Association of Teachers of Mathematics in New England (held in Cambridge), Mr. Charles H. Mergendahl, Miss Harriett R. Pierce, Mr. Joseph L. Powers, Miss Myrtice D. Cheney, and Mr. Ernest G. Hapgood conducted a symposium on the subject, "The College Board Entrance Examination in Elementary Algebra—Can the High Schools Give Sufficient Preparation in the Time Available?"

THE Central Coast Section of the California Teachers' Association met at Monterey, December thirteenth to December sixteenth, inclusive. The mathematics teachers of this section, including some thirty or more teachers, assembled daily for conference. Professor Florian Cajori, from the University of California, was the instructor for the mathematics group, and his work was supplemented by topics given by the teachers and by round-table discussions.

The programs rendered were as follows:

1. "To What Extent Should Mathematics Be Unified in High School Teaching?" Professor Florian Cajori.
2. "Shop Mathematics." Wm. F. Elzinga, Santa Cruz, Calif.
3. "Agriculture Mathematics." S. J. Bensacca, Santa Cruz, Calif.
4. "The Value of Mathematical Recreations and of the History of Mathematics." Professor Florian Cajori.
5. "Reorganization of the Mathematics by a Change in the Subject Matter to Fit the Small High School." Mr. R. P. Binkley, San Juan, Calif.
6. "Segregation of the Mathematics Students into Ability Groups as Exemplified by the Santa Maria High School." J. Calvin Funk, Santa Maria, Calif.
7. "Special Graphs for High School Students." G. M. Weller, King City, Calif.
8. "The Place of Mathematics in the Secondary Schools; Transfer of Mathematical Power to Other Subjects." Professor Florian Cajori.
9. "Commercial Mathematics." G. O. Munson, Watsonville, Calif.
10. "Organization of Mathematics Clubs in the Secondary Schools." Miss Alice Mayberry, Pacific Grove, Calif.

11. "Tests and Measurements of Mathematical Ability." Miss Edith A. Anthony, Pacific Grove, Calif.

The meetings were well well attended, and the discussions valuable. Many suggestions were received from the vocational teachers, and it was decided that mathematics teachers should teach by giving concrete examples whenever possible; but in no case should the teaching of the fundamentals be neglected in order to train the student for some particular vocation. The student must be master of the fundamentals, and these he may apply to the various problems with which he meets. It was further decided that mathematics teachers should give more attention to individual students; each student should be provided with work according to his ability; and as much assistance should be given to the bright student as to the dull student, since it is conceded that upon the bright student depends the destiny of the nation. Many interesting books were recommended by Professor Cajori; general mathematics books for junior high schools were recommended; books classifying work into minimum requirements, "B" work requirements and "A" work requirements, were discussed, but not generally recommended; books on mathematical recreations, fallacies and historical topics were recommended; *Flatland* by E. A. Abbot, *Boy's Own Arithmetic* by Raymond Weeks, *Mathematical Recreations* by W. W. R. Ball were recommended as supplementary books for high school students. Miss Alice Mayberry, from Pacific Grove, presented the group with copies of *Suggested Topics for Mathematics Clubs.*

The mathematics teachers of this section meet once a year at the Central Coast Section of the California Teachers' Association for the discussion of mathematical problems. (Miss Ethel E. Barnebey, San Benite Co. Junior College, Hollister, California.)

THE SECOND YEARBOOK
OF THE
NATIONAL COUNCIL OF TEACHERS OF MATHEMATICS

The Second Yearbook is devoted to Curriculum Problems in Teaching Mathematics. It is divided into three parts as follows: (1) Arithmetic; (2) Junior High School Mathematics; and (3) Senior High School Mathematics.

The contributors are recognized leaders in their fields. Professor F. B. Knight (University of Iowa), Professor G. T. Buswell (University of Chicago), and Miss Jessie P. Haynes (Richmond, Va.) prepared Part I on Arithmetic.

Professor Ralph Beatley (Harvard), Mr. Harry C. Barber (Charlestown High School, Boston), Mr. C. L. Thiele (Assistant Director of Exact Sciences, Detroit), and Professors David Eugene Smith and William D. Reeve (Teachers College) contributed Part II on Junior High School Mathematics.

The curriculum problems of the Senior High School are discussed in Part III by Professor Smith, Miss Gertrude Allen (University High School, Oakland, Calif.), and Mr. E. R. Smith (Headmaster of the Beaver Country Day School, Chestnut Hill, Mass.).

The Yearbook is a distinct contribution to the teaching of mathematics. It was prepared under the direction of a committee, of which Professor W. D. Reeve was Chairman, and is distributed by the Bureau of Publications, Teachers College, New York City, at $1.25 per copy.